

## A Short Proof of the Planarity Characterization of Colin de Verdière

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Colin de Verdière introduced an interesting new invariant  $\mu(G)$  for graphs  $G$ , based on algebraic and analytic properties of matrices associated with  $G$ . He showed that the invariant is monotone under taking minors and moreover, that  $\mu(G) \leq 3$  if and only if  $G$  is planar. In this paper we give a short proof of Colin de Verdière's result that  $\mu(G) \leq 3$  if  $G$  is planar. © 1995 Academic Press, Inc.

### 1. INTRODUCTION

Let  $G$  be a connected undirected graph, which throughout this paper we assume without loss of generality to have vertex set  $\{1, \dots, n\}$ . Then Colin de Verdière's invariant  $\mu(G)$  [3] (English translation: [4]) is the largest corank of any symmetric  $n \times n$  matrix  $M = (m_{i,j})$

$$\begin{aligned} &\text{with exactly one negative eigenvalue (of multiplicity 1)} \\ &\text{and with } m_{i,j} < 0 \text{ if } i \text{ and } j \text{ are adjacent and } m_{i,j} = 0 \text{ if} \\ &i \text{ and } j \text{ are not adjacent and } i \neq j, \end{aligned} \tag{1}$$

so that  $M$  fulfils the "Strong Arnold Hypothesis". (The *corank* of a matrix is the dimension of its kernel.) For the "Strong Arnold Hypothesis" we refer to Colin de Verdière [3]; we do not need it in this paper.

In [3] it is proved that if  $G'$  is a minor of  $G$ , then  $\mu(G') \leq \mu(G)$ . (In proving this, the "Strong Arnold Hypothesis" is essential.) So for each fixed  $t$ , the class of graphs  $G$  satisfying  $\mu(G) \leq t$  is closed under taking minors. Hence, by the theorem of Robertson and Seymour [6], there is a finite collection of "forbidden minors" for such a class of graphs.

Colin de Verdière [3] showed that the graphs  $G$  satisfying  $\mu(G) \leq 1$  are exactly the paths, those satisfying  $\mu(G) \leq 2$  are exactly the outerplanar graphs, and those satisfying  $\mu(G) \leq 3$  are exactly the planar graphs. If  $\mu(G) \leq 4$  then  $G$  is linklessly embeddable, since each graph  $G$  in the

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complete class of forbidden minors found by Robertson, Seymour, and Thomas [7] has  $\mu(G) > 4$  (cf. Bacher and Colin de Verdière [1]). In fact, Robertson, Seymour, and Thomas [8] conjecture that also the reverse implication holds.

Colin de Verdière's proof [3] of the fact that  $\mu(G) \leq 3$  for planar graphs  $G$  uses notions of differential geometry and in particular Cheng's result [2] on the multiplicity of the second eigenvalue of a Laplacian on the 2-sphere. Bacher and Colin de Verdière [1] give a proof that uses the facts that under some conditions  $\mu$  is invariant under  $\Delta Y$ - and  $Y\Delta$ -transformations and that a planar graph can be reduced to an edge by these transformations. We here give a direct combinatorial proof.

## 2. THE PROOF

We first give an auxiliary result. For any vector  $x$ , let  $\text{supp}(x)$  denote the support of  $x$  (i.e., the set  $\{i \mid x_i \neq 0\}$ ). Moreover we denote  $\text{supp}_+(x) := \{i \mid x_i > 0\}$  and  $\text{supp}_-(x) := \{i \mid x_i < 0\}$ . For any subset  $U$  of  $V$  let  $\langle U \rangle$  denote the subgraph of  $G$  induced by  $U$ . If  $x \in \mathbb{R}^n$  and  $I \subseteq \{1, \dots, n\}$ , then  $x_I$  denotes the subvector of  $x$  induced by the indices in  $I$ . Similarly, if  $M$  is an  $n \times n$  matrix and  $I, J \subseteq \{1, \dots, n\}$ , then  $M_{I \times J}$  denotes the submatrix of  $M$  induced by row indices in  $I$  and column indices in  $J$ . We say that a vector  $x \in \ker(M)$  has *minimal support* if  $x$  is nonzero and if for each nonzero vector  $y \in \ker(M)$  with  $\text{supp}(y) \subseteq \text{supp}(x)$  one has  $\text{supp}(y) = \text{supp}(x)$ . Viewing the matrix  $M$  as a Laplacian the following proposition can be regarded as a Courant nodal theorem [5] for graphs.

**PROPOSITION 1.** *Let  $G$  be a connected graph and let  $M$  satisfy (1). Let  $x \in \ker(M)$  have minimal support. Then  $\langle \text{supp}_+(x) \rangle$  and  $\langle \text{supp}_-(x) \rangle$  are connected.*

*Proof.* Suppose for instance that  $\langle \text{supp}_+(x) \rangle$  is disconnected. Let  $I$  and  $J$  be two components of  $\langle \text{supp}_+(x) \rangle$ . Let  $K := \text{supp}_-(x)$ . Since  $m_{i,j} = 0$  if  $i \in I, j \in J$ , we have:

$$\begin{aligned} M_{I \times I} x_I + M_{I \times K} x_K &= 0, \\ M_{J \times J} x_J + M_{J \times K} x_K &= 0. \end{aligned} \tag{2}$$

Let  $z$  be an eigenvector of  $M$  with negative eigenvalue. By the Perron-Frobenius theorem we may assume  $z > 0$ . Let

$$\lambda := \frac{z_I^T x_I}{z_J^T x_J}. \tag{3}$$

Define  $y \in \mathbb{R}^n$  by:  $y_i := x_i$  if  $i \in I$ ,  $y_i := -\lambda x_i$  if  $i \in J$ , and  $x_i := 0$  if  $i \notin I \cup J$ . By (3),  $z^T y = z^T x_I - \lambda z^T x_J = 0$ . Moreover, one has (since  $m_{i,j} = 0$  if  $i \in I$  and  $j \in J$ ):

$$\begin{aligned} y^T M y &= y_I^T M_{I \times I} y_I + y_J^T M_{J \times J} y_J = x_I^T M_{I \times I} x_I + \lambda^2 x_J^T M_{J \times J} x_J \\ &= -x_I^T M_{I \times K} x_K - \lambda^2 x_J^T M_{J \times K} x_K \leq 0, \end{aligned} \tag{4}$$

(using (2)) since  $M_{I \times K}$  and  $M_{J \times K}$  are nonpositive, and since  $x_I > 0$ ,  $x_J > 0$  and  $x_K < 0$ .

Now  $z^T y = 0$  and  $y^T M y \leq 0$  imply that  $M y = 0$  (as  $M$  is symmetric and has exactly one negative eigenvalue, with eigenvector  $z$ ). Therefore,  $y \in \ker(M)$ , contradicting the minimality of  $\text{supp}(x)$ . ■

From this we derive:

**THEOREM 1.** *If  $G$  is planar then  $\mu(G) \leq 3$ .*

*Proof.* Since  $\mu(G)$  does not increase after deleting edges, we may assume that  $G$  is maximally planar. So  $G$  is 3-connected and contains a triangle which is a face. Let  $U$  be the set of vertices of this triangle. Assume that  $\mu(G) > 3$ . Let  $M = (m_{i,j})$  be a matrix satisfying (1) with corank equal to  $\mu(G)$ . Since the corank of  $M$  is larger than 3,  $\ker(M)$  contains a nonzero vector  $x$  with  $x_i = 0$  for all  $i \in U$ . We may assume that  $x$  has minimal support.

Since  $G$  is 3-connected there exist 3 pairwise disjoint paths  $P_1, P_2, P_3$ , where each  $P_i$  starts in a vertex  $v_i \notin \text{supp}(x)$  adjacent to at least one vertex in  $\text{supp}(x)$ , and ends in  $U$ . Now if  $M$  satisfies (1), then each vertex  $v \notin \text{supp}(x)$  adjacent to some vertex in  $\text{supp}_+(x)$  is also adjacent to some vertex in  $\text{supp}_-(x)$  and conversely. So each  $v_i$  is adjacent to at least one vertex in  $\text{supp}_+(x)$  and at least one vertex in  $\text{supp}_-(x)$ .

By Proposition 1,  $\text{supp}_+(x)$  and  $\text{supp}_-(x)$  can be contracted to one vertex each. Delete all vertices of  $G$  not contained in  $\text{supp}(x)$  or in any  $P_i$  and contract each  $P_i$  to one vertex. Add a vertex inside the triangle and edges between this vertex and the vertices in  $U$ . The graph we obtain is still planar. But this graph contains  $K_{3,3}$  as subgraph, hence we have a contradiction. ■

The proof of this theorem is similar to the proof given in Cheng [2] for the maximal multiplicity of the second eigenvalue of a Laplacian on the 2-sphere. We have chosen a nonzero vector vanishing on the vertices of a triangle and then used a Courant nodal theorem for graphs to obtain a contradiction. Cheng showed that the multiplicity of the second eigenvalue of a Laplacian on the 2-sphere is at most 3, by choosing an eigenfunction whose value and both partial derivatives vanish at some given point of the

2-sphere. But the positive or negative support of this eigenfunction consists of more than one component, contradicting Courant nodal theorem.

Finally we mention the following corollary of Proposition 1.

**COROLLARY 1.** *Let  $G$  be a connected graph and let  $M$  satisfy (1). If  $\ker(M)$  has dimension 1 and  $x \in \ker(M)$  then  $\langle \text{supp}_+(x) \rangle$  and  $\langle \text{supp}_-(x) \rangle$  are connected.*

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